Revision Sheet

Partial derivatives

(1) If $w = f(x, y)$ and $x = r\cos\theta,\ y = r\sin\theta$ Show that

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2$$

(2) If $w = \ln(x^2 + y^2)$ Show that $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 2$

(3) If $w = \sin^{-1}(x^2 + y^2)$ Show that $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 2\tan w$

(4) If $w = \tan^{-1}(x^3 + y^3)$ Show that $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 3\sin w\cos w$

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(5) Find the general solution of the following differential equations:

$(a)\ x(1+y^2)dx + y(1+x^2)dy = 0$

$(b)\ x\sqrt{1-y^2}dx + y\sqrt{1-x^2}dy = 0$

$(c)\ xydx + \sqrt{1+x^2}dy = 0$

$(d)\ y' + \frac{y}{x} = \sin x$

$(e)\ y' + y\tan x = \cos^3 x$

$(f)\ y' + y\cot x = \sin^2 x$

$(g)\ (D^2 + 9)y = \cos 2x + \sin 2x$

$(h)\ (D^2 + 1)y = \sin x \sin 2x$

$(i)\ (D^2 - 6D + 13)y = 8e^{3x}\sin 2x$

(6) Evaluate $\oint_C \vec{F}.d\vec{r}$ where $\vec{F}(t) = (x - 3y)\vec{i} + (y - 2x)\vec{j}$ and C is the closed curve $x = 3\sin\theta,\ y = 2\cos\theta$ in xy – plane [Ans.: 6π].

(7) Evaluate $\oint_C \vec{F}.d\vec{r}$ where $\vec{F}(t) = (2x + y^2)\vec{i} + (3y - 4x)\vec{j}$ and C is the closed

(8) Show that the following functions satisfy Cauchy Riemann equations

$(a)\ f(z) = iz^2 + 2z \qquad (b)\ f(z) = \sin z \qquad (c)\ f(z) = ze^{-z}$

(9) Prove that **(i)** $\overline{\sin z} = \sin\bar{z}$ **(ii)** $\overline{\cos z} = \cos\bar{z}$

(10) By using Cauchy integral Formula evaluate the following integrals:

(a) $\oint_C \frac{z^2+5}{z-3} dz$ where C is the circle $|z| = 4$.

(b) $\oint_C \frac{z\,dz}{z^2-1}$ where C is the circle $|z| = 2$. (Ans.$2\pi i$)

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Solved examples

Example (1): Find the general solution of the following differential equations:

$$xydx + (x^2+1)dy = 0$$

Solution:

Divided the equation by $y(x^2+1)$

$\frac{x}{(x^2+1)}dx + \frac{dy}{y} = 0$ The variables are separated, integrate the equation

$$\int \frac{x}{(x^2+1)}dx + \int \frac{dy}{y} = c$$

$$\frac{1}{2}\ln(x^2+1) + \ln y = \ln c$$

$$\ln(x^2+1)^{\frac{1}{2}} + \ln y = \ln c$$

$$\ln\sqrt{(x^2+1)} + \ln y = c \rightarrow \ln y\sqrt{(x^2+1)} = \ln c \rightarrow y\sqrt{(x^2+1)} = c$$

The solution of the equation is $\boxed{y^2x^2 + y^2 = C}$

Another solution

$$xydx + (x^2+1)dy = 0$$

$$xydx + (x^2dy + dy) = 0$$

$$x(ydx + xdy) + dy = 0$$

$$xd(xy) + dy = 0$$

$$(xy)d(xy) + ydy = 0$$

Integrate

$$\frac{1}{2}(xy)^2+\frac{1}{2}y^2=c$$

$$\boxed{x^2y^2+y^2=C}$$

Example (2):

Find the general solution of the differential equations: $(1+y^2)dx+(1+x^2)dy=0$

Solution:

divided the equation by $(1+y^2)(1+x^2)$

$\frac{1}{(1+x^2)}dx+\frac{1}{(1+y^2)}dy=0$ Integrate the equation we get $\tan^{-1}x+\tan^{-1}y=C$

Example (3):

Find the general solution of the differential equations: (3) $\sqrt{1-y^2}dx+\sqrt{1-x^2}dy=0$

Solution:

divided the equation by $\sqrt{1-y^2}\sqrt{1-x^2}$

$$\frac{1}{\sqrt{1-x^2}}dx+\frac{1}{\sqrt{1-y^2}}dy=0$$

Integrate the equation. $\int\frac{1}{\sqrt{1-x^2}}dx+\int\frac{1}{\sqrt{1-y^2}}dy=c$ then $\sin^{-1}x+\sin^{-1}y=c$

Example (4):

Find the general solution of the differential equations: $y'\tan x-y=1$

Solution:

The equation can be written in the form

$$\tan x\frac{dy}{dx}-y=1\Rightarrow \tan x dy=(y+1)dx$$

divided by $(y+1)\tan x$

$$\frac{dy}{(y+1)}=\frac{1}{\tan x}dx=\frac{\cos x}{\sin x}dx \qquad \therefore\int\frac{dy}{(y+1)}=\int\frac{\cos x}{\sin x}dx$$

$\ln(y+1)=\ln \operatorname{cosec}x+\ln c$ or $y+1=C\operatorname{cosec}x$

Example (5):

Find the general solution of the differential equations: $xydx + \sqrt{1+x^2}dy = 0$

Solution:

divided by $y\sqrt{1+x^2}$

$$\frac{xy}{y\sqrt{1+x^2}}dx + \frac{\sqrt{1+x^2}}{y\sqrt{1+x^2}}dy = 0$$

$$\frac{x}{\sqrt{1+x^2}}dx + \frac{1}{y}dy = 0$$

$$\frac{1}{2}\int \frac{2x}{\sqrt{1+x^2}}dx + \int \frac{1}{y}dy = c$$

$$\sqrt{1+x^2} + \ln y = c$$

Example (6):

Solve the differential equation $xdy - ydx = \sqrt{x^2 - y^2}\, dx$

Solution:

$M(x,y), N(x,y)$ are homogeneous of the same degree(first degree)

let $y = ux \quad \therefore dy = udx + xdu$

substitute in the differential equation we have

$$x(udx + xdy) - uxdx = \sqrt{x^2 - u^2x^2}\, dx$$

$$x(udx + xdy) - uxdx = x\sqrt{1-u^2}\, dx$$

divided by **x and** separate the variable

$$(udx + xdu) - udx = \sqrt{1-u^2}\, dx$$

$$xdu = \sqrt{1-u^2}\, dx$$

$$\frac{du}{\sqrt{1-u^2}} = \frac{1}{x}dx$$

Integrate the last equation $\int \frac{du}{\sqrt{1-u^2}} = \int \frac{1}{x} dx$

$$\sin^{-1} u = \ln x + \ln C = \ln Cx$$

$\sin^{-1} \frac{y}{x} = \ln Cx \quad \rightarrow \frac{y}{x} = \sin(\ln Cx)$ and the solution is $\boxed{y = x\sin(\ln Cx)}$

<u>Another solution</u>

$$xdy - ydx = \sqrt{x^2 - y^2}\ dx$$

$$\frac{xdy - ydx}{x^2} = \frac{\sqrt{x^2 - y^2}}{x^2} dx$$

$$\frac{xdy - ydx}{x} = \frac{\sqrt{1 - y^2/x^2}}{x} dx$$

$$d\frac{y}{x} = \sqrt{1 - y^2/x^2}\ \frac{dx}{x}$$

Pu $u = y/x$

$$du = \sqrt{1-u^2}\ \frac{dx}{x}$$

$$\frac{du}{\sqrt{1-u^2}} = \frac{dx}{x}$$

By integration

$$\sin^{-1} u = \ln x + \ln c$$

$$\sin^{-1} u = \ln cx$$

$$\boxed{y = x\sin(cx)}$$

<u>Example (7):</u>

Solve the differential equation $(x^2 + y^2)dx = xydy$

<u>solution:</u>

$M(x,y), N(x,y)$ are homogeneous of the same degree(second degree)

let $y = ux \quad \therefore dy = udx + xdu$

substitute in the differential equation we have

$$(x^2+u^2x^2)dx = xux(udx+xdu)$$
$$x^2(1+u^2)dx = x^2u(udx+xdu)$$

divided by x^2 and separate the variable

$$(1+u^2)dx = u(udx+xdu)$$
$$(1+u^2-u^2)dx = uxdu$$
$$\frac{dx}{x} = udu \qquad \therefore \int \frac{dx}{x} = \int udu$$
$$\ln x - \ln c = \frac{1}{2}u^2 \Rightarrow x^2 = ce^{u^2} \qquad \rightarrow \boxed{x^2 = ce^{y^2/x^2}}$$

Another solution

$$(x^2+y^2)dx = xydy$$
$$(x^2+y^2)dx - xydy = 0$$
$$x^2dx + y(ydx - xdy) = 0$$
$$\frac{x^2}{x^3}dx + \frac{y}{x}\frac{(ydx-xdy)}{x^2} = 0$$
$$\frac{1}{x}dx - \left(\frac{y}{x}\right)\frac{(xdy-ydx)}{x^2} = 0$$
$$\frac{1}{x}dx - \left(\frac{y}{x}\right)d\left(\frac{y}{x}\right) = 0$$
$$\ln x - \ln C - \frac{1}{2}\left(\frac{y}{x}\right)^2 = 0 \qquad \boxed{x^2 = Ae^{y^2/x^2}}$$

Example (8):

Solve the differential equation $(xy - x^2)dy - y^2dx = 0$

solution:

$M(x,y), N(x,y)$ are homogeneous of the same degree(second degree)

let $y = ux \quad \therefore dy = udx + xdu$

substitute in the differential equation we have

$$(x^2u - x^2)(udx + xdu) - x^2u^2dx = 0$$
$$x^2(u-1)(udx+xdu) - x^2u^2dx = 0$$

Divided by x^2 and separate the variable

$$(u-1)(udx+xdu)-u^2dx=0$$

$$(u-1)udx+(u-1)xdu-u^2dx=0$$

$$\left[(u-1)u-u^2\right]dx+(u-1)xdu=0$$

$$-udx+(u-1)xdu=0$$

$$-\frac{dx}{x}+\frac{(u-1)}{u}du=0$$

$-\frac{dx}{x}+(1-\frac{1}{u})du=0$ by integration we have $-\ln x+u-\ln u=\ln c$

$$\ln cxu=u \qquad \Rightarrow \boxed{\ln cy=\frac{y}{x}}$$

Another solution

$$(xy-x^2)dy-y^2dx=0 \quad \Rightarrow \quad xydy-x^2dy-y^2dx=0$$

$$xydy-y^2dx-x^2dy=0$$

$$y(xdy-ydx)-x^2dy=0 \quad \Rightarrow \quad \frac{(xdy-ydx)}{x^2}-\frac{1}{y}dy=0$$

$d(\frac{y}{x})-\frac{1}{y}dy=0$ integrate $\boxed{\frac{y}{x}-\ln y=\ln C}$

$$\frac{y}{x}=\ln C+\ln y=\ln Cy \quad \Rightarrow \boxed{y=xe^{Cy}}$$

Example (9): Solve $y'+\frac{y}{x}=\sin x$

Solution:

The equation is a linear and $P(x)=\frac{1}{x}$, $Q(x)=\sin x$. We can determine the integrating factor

$$\int P(x)dx=\int \frac{dx}{x}=\ln x \qquad \therefore \mu=e^{\ln x}=x$$

The general solution is

$$y\ x=\int x\sin x\,dx+C=-x\cos x-\sin x+C$$

$$y=\frac{-x\cos x-\sin x+C}{x}$$

Example (10)

Solve the following differential equation

(a) $y' + y\tan x = \sin x$

Answer

The equation is a linear and $P(x) = \tan x,\ \ Q(x) = \sin x$

We can determine the integrating factor

$\int P(x)dx = \int \tan x dx = \ln \sec x$

$\therefore \mu = e^{\ln \sec x} = \sec x$

multiply the equation by $\sec x$

$y' \sec x + y \sec x \tan x = \sec x \sin x = \tan x$

which is exact differential equation

and the left side is the derivative of $y \sec x$

$\therefore d(y\sec x) = \tan x$

by integration

$y\sec x = \int \tan x dx = \sec x + C$

$$\boxed{y\sec x = \sec x + C}$$

Example (11):

Solve $y' + y\tan x = \cos^3 x$

Solution:

The equation is a linear and $P(x) = \tan x,\ \ Q(x) = \cos^3 x$. We can determine the integrating factor

$\int P(x)dx = \int \tan x dx = \ln \sec x \qquad \therefore \mu = e^{\ln \sec x} = \sec x$

multiply the equation by $\sec x \qquad y' \sec x + y \sec x \tan x = \sec x \cos^3 x = \cos^2 x$

which is exact differential equation and the left side is the derivative of $y \sec x$

$\therefore d(y\sec x) = \cos^2 x \qquad$ by integration

$$y\sec x=\int \cos^2 x dx=\frac{1}{2}\int(1+\cos 2x)dx=\frac{1}{2}[x-\frac{1}{2}\sin 2x]+C$$

$$\therefore y=\frac{1}{2}x\cos x-\frac{1}{4}\sin 2x\cos x+C\cos x=\frac{1}{2}x\cos x-\frac{1}{2}\sin x\cos^2 x+C\cos x$$

is the general solution of the given equation.

Example (12): Solve $y'+y\cot x=\sin^2 x$

Solution:

The equation is a linear and $P(x)=\cot x,\quad Q(x)=\sin^2 x$

We can determine the integrating factor

$$\int P(x)dx=\int \cot x dx=\ln\sin x$$

$$\therefore \mu=e^{\ln\sin x}=\sin x$$

multiply the equation by $\sin x$

$$y'\sin x+y\sin x\cot x=\sin^3 x$$

$$y'\sin x+y\cos x=\sin^3 x$$

Which is exact differential equation and the left side is the derivative of $y\sin x$

$\therefore d(y\sin x)=\sin^3 x$ and by integration

$$y\sin x=\int \sin^3 x\, dx=\int\sin^2 x\sin x\, dx=\int(1-\cos^2 x)\sin x\, dx=\int(\sin x dx-\cos^2 x\sin x\, dx)$$

$$y\sin x=-\cos x+\frac{1}{3}\cos^3 x+C$$

$$\therefore y=-\cot x+\frac{1}{3}\frac{\cos^2 x}{\sin x}+\frac{C}{\sin x}$$

is the general solution of the given equation.

Example (13): Solve $xdx+(y+yx^2+y^3)dy=0$

Solution:

we can rewrite the equation in the form

$$x\,dx+y\,dy+yx^2dy+y^3dy=0$$

$$(x\,dx+y\,dy)+y(x^2+y^2)dy=0$$

$$\frac{(x\,dx + y\,dy)}{(x^2+y^2)} + y\,dy = 0 \qquad \frac{1}{2}\frac{d(x^2+y^2)}{(x^2+y^2)} + y dy = 0$$

the last equation can be integrated $\frac{1}{2}\ln(x^2+y^2)+\frac{1}{2}y^2 = C \rightarrow \ln(x^2+y^2)+y^2 = A$

<u>Example (14):</u> Solve $(D^2-3D+2)y=0$

<u>Solution</u>

The characteristic equation in the form $m^2-3m+2=0$ which gives roots $m_1=1, m_2=2$ (Distinct roots). Hence $y_{C.F.}=C_1e^x+C_2e^{2x}$.

Where C_1 *and* C_2 are arbitrary constants. Since $g(x)=0$, so general solution is

$y_{G.S.}=y_{C.F.}=C_1e^x+C_2e^{2x}$.

<u>Example (15)</u> $(D^2-5D+6)y=e^x\sinh 6x$

<u>Solution:</u>

The characteristic equation is

$m^2-5m+6=0$ then $(m-2)(m-3)=0 \rightarrow m=2,3$ and $y_c=C_1e^{3x}+C_2e^{2x}$

and

$$y_p=\frac{1}{D^2-5D+6}e^x\sinh 6x = e^x\frac{1}{(D+1)^2-5(D+1)+6}\sinh 2x$$

$$=e^x\frac{1}{D^2+2D+1-5D-5+6}\sinh 6x = e^x\frac{1}{D^2-3D+2}\sinh 6x$$

$$=e^x\frac{1}{6^2-3D+2}\sinh 6x = e^x\frac{1}{38-3D}\sinh 6x = e^x\frac{(38-3D)}{38^2-9D^2}\sinh 6x$$

$$=\frac{e^x}{38^2-9(6)^2}(38\sinh 6x-18\cosh 6x)$$

Then the general solution in the form

$$\boxed{y_G=C_1e^{3x}+C_2e^{2x}+\frac{e^x}{38^2-9(6)^2}(38\sinh 6x-18\cosh 6x)}$$

<u>Another solution</u>

$$y_p=\frac{1}{D^2-5D+6}e^x\sinh 6x=\frac{1}{2}\frac{1}{D^2-5D+6}e^x\left(e^{6x}-e^{-6x}\right)$$

$$=\frac{1}{2}\frac{1}{D^2-5D+6}\left(e^{7x}-e^{-5x}\right)=\frac{1}{2}\left(\frac{1}{D^2-5D+6}e^{7x}-\frac{1}{D^2-5D+6}e^{-5x}\right)$$

$$=\frac{1}{2}\left(\frac{1}{7^2-5(7)+6}e^{7x}-\frac{1}{(-5)^2-5(5)+6}e^{-5x}\right)=\frac{1}{2}\left(\frac{1}{20}e^{7x}-\frac{1}{6}e^{-5x}\right)$$

Then the general solution in the form

$$\boxed{y_G=C_1e^{3x}+C_2e^{2x}+\frac{1}{2}\left(\frac{1}{20}e^{7x}-\frac{1}{6}e^{-5x}\right)}$$

Example (16): Solve $(D^2+4)y=\sin^2 x$.

Solution

The characteristic equation is $m^2+4=0$

then $m_1,m_2=\pm 2i$ and $y_C=C_1\cos 2x+C_2\sin 2x$

$$y_p=\frac{1}{(D^2+4)}\sin^2 x=\frac{1}{(D^2+4)}\left(\frac{1}{2}\right)(1-\cos 2x)$$

$$=\frac{1}{2}\frac{1}{(D^2+4)}(1)-\frac{1}{(D^2+4)}\cos 2x \;=\frac{1}{2}\left(\frac{1}{(D^2+4)}e^{0.x}-\frac{x\sin 2x}{4}\right)$$

$$=\frac{1}{2}\left(\frac{1}{(0+4)}e^{0.x}-\frac{x\sin 2x}{4}\right)=\frac{1}{2}\left(\frac{1}{4}-\frac{x\sin 2x}{4}\right)=\left(\frac{1}{8}-\frac{x\sin 2x}{8}\right)$$

The general solution in the form $y_G=C_1\cos 2x+C_2\sin 2x+\frac{1}{8}-\frac{x\sin 2x}{8}$

Vector Analysis

Example (17): If $\vec{A}=x^2z\vec{i}-2y^3z^2\vec{j}+xy^2z\vec{k}$ find $\vec{\nabla}.\vec{A}$.

Solution:

$$\vec{\nabla}.\vec{A}=div\,\vec{A}=\left(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}\right).(x^2z\vec{i}-2y^3z^2\vec{j}+xy^2z\;\vec{k})$$

$$=\frac{\partial}{\partial x}x^2z+\frac{\partial}{\partial y}(-2y^3z^2)+\frac{\partial}{\partial z}xy^2z \;=2xz-6y^2z^2+xy^2.$$

Example (18): If $\phi=2x^3y^2z^4$ find $\vec{\nabla}.\vec{\nabla}\phi$.

Solution:

$$\vec{\nabla}\phi=\vec{\nabla}(2x^3y^2z^4)=\frac{\partial}{\partial x}(2x^3y^2z^4)\vec{i}+\frac{\partial}{\partial y}(2x^3y^2z^4)\vec{j}+\frac{\partial}{\partial z}(2x^3y^2z^4)\vec{k}$$

$$=6x^2y^2z^4\vec{i}+4x^3yz^4\vec{j}+8x^3y^2z^3\vec{k}$$

$$\vec{\nabla}.\vec{\nabla}\phi=(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}).(6x^2y^2z^4\vec{i}+4x^3y\ z^4\ \vec{j}+8x^3y^2z^3\ \vec{k})$$

$$=\frac{\partial}{\partial x}6x^2y^2z^4+\frac{\partial}{\partial y}4x^3y\ z^4+\frac{\partial}{\partial z}8x^3y^2z^3=12xy^2z^4+4x^3z^4+24x^3y^2z^2.$$

Example (19):

find $\vec{\nabla}.\ \vec{\nabla}\frac{1}{|\vec{r}|}$ .where $\vec{r}$ is the position vector for any point (x,y,z) in the space.

Solution:

$$\vec{\nabla}\frac{1}{|\vec{r}|}=\vec{\nabla}(x^2+y^2+z^2)^{\frac{-1}{2}}=\frac{-1}{2}(x^2+y^2+z^2)^{\frac{-3}{2}}\ \vec{\nabla}(x^2+y^2+z^2)$$

$$=\frac{-1}{2}(x^2+y^2+z^2)^{\frac{-3}{2}}[\frac{\partial}{\partial x}(x^2+y^2+z^2)\vec{i}+\frac{\partial}{\partial y}(x^2+y^2+z^2)\vec{j}+\frac{\partial}{\partial z}(x^2+y^2+z^2)\vec{k}]$$

$$\vec{\nabla}\frac{1}{|\vec{r}|}=\frac{-1}{2}(x^2+y^2+z^2)^{\frac{-3}{2}}[2x\vec{i}+2y\vec{j}+2z\vec{k}]$$

$$=-(x^2+y^2+z^2)^{\frac{-3}{2}}(x\vec{i}+y\vec{j}+z\vec{k})=\frac{-(x\vec{i}+y\vec{j}+z\vec{k})}{(x^2+y^2+z^2)^{\frac{3}{2}}}=\frac{-\vec{r}}{|\vec{r}|^3}$$

Example (20):

If $\vec{A}=x^2y\ \vec{i}-2xz\ \vec{j}+2yz\ \ \vec{k}$ Find $\vec{\nabla}\times(\vec{\nabla}\times\vec{A})$

Solution

$$(\vec{\nabla}\times\vec{A})=[(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k})\times(x^2y\vec{i}-2xz\vec{j}+2yz\vec{k})]=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2y & -2xz & 2yz\end{vmatrix}$$

$$=[(\frac{\partial}{\partial y}(2yz)-\frac{\partial}{\partial z}(-2xz)]\ \vec{i}+[(\frac{\partial}{\partial z}(x^2y)-\frac{\partial}{\partial x}(2yz)]\ \vec{j}+[(\frac{\partial}{\partial x}(-2xz)-\frac{\partial}{\partial y}(x^2y)]\vec{k}=(2x+2z)\ \vec{i}-(x^2+2z)\ \vec{k}$$

$$\therefore \vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\vec{\nabla}\times[(2x+2z)\,\vec{i}-(x^2+2z)\,\vec{k}]=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2x+2z & 0 & -(x^2+2z)\end{vmatrix}=(2x+2)\vec{j}$$

Example (21):

If $\vec{F}=x^2yz\,\vec{i}+xyz\,\vec{j}-xyz^2\,\vec{k}$ Find $div\ \vec{F},\quad curl\ \vec{F}$

Solution:

$$div\ \vec{F}=\nabla\cdot\vec{F}=\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}$$

$$=\frac{\partial}{\partial x}\left(x^2yz\right)+\frac{\partial}{\partial y}(xyz)+\frac{\partial}{\partial z}\left(-xyz^2\right)=2xyz+xz-2xyz=xz$$

$$curl\ \vec{F}=\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)\vec{i}+\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)\vec{j}+\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\vec{k}$$

$$=\left[\frac{\partial}{\partial y}\left(-xyz^2\right)-\frac{\partial}{\partial z}(xyz)\right]\vec{i}+\left[\frac{\partial}{\partial z}\left(x^2yz\right)-\frac{\partial}{\partial x}\left(-xyz^2\right)\right]\vec{j}+\left[\frac{\partial}{\partial x}(xyz)-\frac{\partial}{\partial y}\left(x^2yz\right)\right]\vec{k}$$

$$=\left[\left(-xz^2\right)-(xy)\right]\vec{i}+\left[\left(x^2y\right)-\left(-yz^2\right)\right]\vec{j}+\left[(yz)-\left(x^2z\right)\right]\vec{k}$$

$$=\left(-xz^2-xy\right)\vec{i}+\left(x^2y+yz^2\right)\vec{j}+\left(yz-x^2z\right)\vec{k}$$

Example (22): Show that $\vec{\nabla}\times\vec{\nabla}\phi=0$

Solution

$$\vec{\nabla}\times(\vec{\nabla}\phi)=\vec{\nabla}\times\left(\frac{\partial\phi}{\partial x}\vec{i}+\frac{\partial\phi}{\partial y}\vec{j}+\frac{\partial\phi}{\partial z}\vec{k}\right)=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z}\end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right)-\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial y}\right)\right]\vec{i}+\left[\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial x}\right)-\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial z}\right)\right]\vec{j}+\left[\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right)-\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right)\right]\vec{k}$$

$$=\left[\frac{\partial^2\phi}{\partial y\partial z}-\frac{\partial^2\phi}{\partial z\partial y}\right]\vec{i}+\left[\frac{\partial^2\phi}{\partial z\partial x}-\frac{\partial^2\phi}{\partial x\partial z}\right]\vec{j}+\left[\frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial^2\phi}{\partial y\partial x}\right]\vec{k}=0$$

Example (23): Show that $\vec{\nabla}.(\vec{\nabla}\times\vec{A})=0$

let $\vec{A}=a_x\vec{i}+a_y\vec{j}+a_z\vec{k}$

$$(ii)\ \vec{\nabla}.(\vec{\nabla}\times\vec{A})=\vec{\nabla}.[(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k})\times(a_x\vec{i}+a_y\vec{j}+a_z\vec{k})]$$

$$\therefore\vec{\nabla}.(\vec{\nabla}\times\vec{A})=\vec{\nabla}.[(\frac{\partial a_z}{\partial y}-\frac{\partial a_y}{\partial z})\vec{i}+(\frac{\partial a_x}{\partial z}-\frac{\partial a_z}{\partial x})\vec{j}+(\frac{\partial a_y}{\partial x}-\frac{\partial a_x}{\partial y})\vec{k}]$$

$$=(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}).[(\frac{\partial a_z}{\partial y}-\frac{\partial a_y}{\partial z})\vec{i}+(\frac{\partial a_x}{\partial z}-\frac{\partial a_z}{\partial x})\vec{j}+(\frac{\partial a_y}{\partial x}-\frac{\partial a_x}{\partial y})\vec{k}]$$

$$=\frac{\partial}{\partial x}(\frac{\partial a_z}{\partial y}-\frac{\partial a_y}{\partial z})+\frac{\partial}{\partial y}(\frac{\partial a_x}{\partial z}-\frac{\partial a_z}{\partial x})+\frac{\partial}{\partial z}(\frac{\partial a_y}{\partial x}-\frac{\partial a_x}{\partial y})$$

$$=(\frac{\partial^2 a_z}{\partial x\partial y}-\frac{\partial a_y}{\partial x\partial z})+(\frac{\partial^2 a_x}{\partial y\partial z}-\frac{\partial^2 a_z}{\partial y\partial x})+(\frac{\partial^2 a_y}{\partial z\partial x}-\frac{\partial^2 a_x}{\partial z\partial y})=0$$

Complex Variable

Example (24): Show that the function $f(x)=z^3$ satisfy Cauchy Riemann Equations

Solution: We note that

$$f(z)=z^3=(x+iy)^3=(x^3-3xy^2)+i(3x^2y-y^3)$$

$\therefore u(x,y)=x^3-3xy^2,\qquad v(x,y)=3x^2y-y^3$

$\therefore u_x=3x^2-3y^2,\qquad v_x=6xy$

$u_y=-6xy,\qquad vy=3x^2-3y^2$

$\therefore u_x=v_y,\qquad u_y=-v_x$

Then Cauchy Riemann equations are satisfied and

$$f'(z) = u_x + iv_x = 3x^2 - 3y^2 + i(6xy)$$
$$= 3[(x^2 - y^2) + i(2xy)] = 3(x + iy)^2 = 3z^2$$

Example (25)

Show that the function $f(z) = \sin z$ Satisfy Cauchy Remman's Equation

Answer

$$\sin z = \sin(x + iy) = \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i \cos x \sinh y$$

$u = \sin x \cosh y \qquad v = \cos x \sinh y$

$\frac{\partial u}{\partial x} = \cos x \cosh y \qquad \frac{\partial v}{\partial x} = -\sin x \sinh y$

$\frac{\partial u}{\partial y} = \sin x \sinh y \qquad \frac{\partial v}{\partial y} = \cos x \cosh y$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \cos x \cosh y \quad and \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \sin x \sinh y$$

Example (26):

Given function $u = e^x \sin y$ Find v such that $f(z) = u + iv$ satisfy Cauchy-Riemann equations.

Solution:

$(i)\ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad (ii)\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

integrate **(i) with respect to** y we have

$$v = \int \frac{\partial u}{\partial x}\, dy + f(x) = \int e^x \sin y\, dy + f(x) = -e^x \cos y + f(x)$$

where $f(x)$ is the integration constant and to determine $f(x)$ we use (ii) as follows

$$\frac{\partial v}{\partial x} = -e^x \cos y + f'(x) = -\frac{\partial u}{\partial y} = -e^x \cos y$$

$\therefore\ f' = 0 \Rightarrow f(x) = C$ (pure arbitrary constant)

Hence $\quad v = -e^x \cos y + C$ and

$$f(z) = u + iv = e^x \sin y - ie^x \cos y + C = -ie^x(\cos y + i \sin y) + C = -ie^z + C$$

where C is an arbitrary constant.

Example (27): Show that $\overline{\sin z} = \sin \bar{z}$.

Solution:

$$\overline{\sin z} = \overline{\sin(x+iy)}$$
$$= \overline{\sin x \cos(iy) + \cos x \sin(iy)} = \overline{\sin x \cosh y + i \cos x \sinh y}$$
$$= \sin x \cosh y - i \cos x \sinh y = \sin x \cos(iy) - \cos x \sin(iy)$$
$$= \sin(x - iy) = \sin \bar{z} \qquad \text{then } \overline{\sin z} = \sin \bar{z}$$

Example(28) Evaluate $\int_{(3.0)}^{(0,3)} (2\bar{z} - z^2)dz$ on the circle $|z| = 3$.

Answer

Use the exponential form for the complex number

$z = 3e^{i\theta}$ then $d\theta = 3e^{i\theta}d\theta$ *and* $\bar{z} = 3e^{-i\theta}$

$$\int_{(3.0)}^{(0,3)} (2\bar{z} - z^2)dz = \int_0^{2\pi} (6e^{-i\theta} - 9e^{2i\theta})3ie^{i\theta}d\theta = \int_0^{2\pi} i(18 - 27e^{3i\theta})d\theta$$
$$= i\left[18\theta - 9e^{3i\theta}\right]_0^{2\pi} = i\left[36\pi - 9e^{6i\pi} + 9\right] = 36\pi i$$

Example (29): Evaluate $\oint_C \frac{z^3 - 6}{2z - i} dz$ where C is the circle $|z| = 2$

Solution: $\oint_C \frac{z^3 - 6}{2z - i} dz = \frac{1}{2} \oint_C \frac{z^3 - 6}{z - \frac{1}{2}i} dz$

Since $\frac{i}{2}$ inside the circle $|z| = 2$ and $\frac{f(z)}{z - z_0} = \frac{z^3 - 6}{z - \frac{1}{2}i}$ then $z_0 = \frac{1}{2}i$,

$f(z) = z^3 - 6$, which is analytic inside and on C and $f(z_0) = (\frac{1}{2}i)^3 - 6 = \frac{-i}{8} - 6$

by using Cauchy's integral Formula

$$\oint_C \frac{f(z)}{z - z_0} dz = 2\pi i\, f(z_0) \quad \text{then} \quad \oint_C \frac{z^3 - 6}{2z - i} dz = \frac{1}{2} \oint_C \frac{z^3 - 6}{z - \frac{1}{2}i} dz = \frac{1}{2}(2\pi i) f(\tfrac{1}{2}i) = \frac{\pi}{8} - 6\pi i$$

Infinite Series

Example (30):

Use ratio test to test the following series for convergence

(i) $\sum_{n=1}^{\infty} \frac{1}{n.3^n}$ **(ii)** $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ **(iii)** $\sum_{n=1}^{\infty} \frac{2^n}{n}$ **(iv)** $\sum_{n=1}^{\infty} \frac{n}{n+1}$

Solution:

(i) $\because a_n = \frac{1}{n.3^n} \quad \therefore a_{n+1} = \frac{1}{(n+1).3^{n+1}}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} [\frac{1}{(n+1).3^{n+1}} \div \frac{1}{n.3^n}] = \lim_{n\to\infty} \frac{n.3^n}{(n+1).3^{n+1}} = \frac{1}{3} < 1$$

then $\sum_{n=1}^{\infty} \frac{1}{n.3^n}$ is converge.

(ii) $\because a_n = \frac{2^n}{n!} \quad \therefore a_{n+1} = \frac{2^{n+1}}{(n+1)!}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} [\frac{2^{n+1}}{(n+1)!} \div \frac{2^n}{n!}]$$

$$= \lim_{n\to\infty} \frac{2^{n+1}.n!}{2^n.(n+1)!} = \lim_{n\to\infty} \frac{2}{(n+1)} = 0 < 1$$

then $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ is converge.

(iii) $\because a_n = \frac{2^n}{n} \quad \therefore a_{n+1} = \frac{2^{n+1}}{(n+1)}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} [\frac{2^{n+1}}{(n+1)} \div \frac{2^n}{n}]$$

$$= \lim_{n\to\infty} \frac{2^{n+1}.n}{2^n.(n+1)} = \lim_{n\to\infty} \frac{2n}{(n+1)} = 2 > 1$$

then $\sum_{n=1}^{\infty} \frac{2^n}{n}$ is diverge.

(iv) $a_n = \dfrac{n}{n+1}, \; a_{n+1} = \dfrac{n+1}{n+2} \qquad \lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \dfrac{n+1}{n+2} \cdot \dfrac{n+1}{n} = 1$

The Ratio test is inconclusive.

<u>Example (32):</u>

Use integral test to test the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2+1}$ for convergence

<u>Solution</u>:

Since $f(x) = \dfrac{1}{x^2+1}$ positive and non-increasing function for all $x > 0$ then we have

$$\int_1^{\infty} \frac{1}{x^2+1} dx = \lim_{a\to\infty} \int_1^{a} \frac{1}{x^2+1} dx = \lim_{a\to\infty} \tan^{-1} x \Big|_1^a$$

$$= \lim_{a\to\infty} \left\{ \left(\tan^{-1} a\right) - \left(\tan^{-1} 1\right) \right\} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

which show that the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2+1}$ is convergent.

<u>Example (33)</u>

(a) Test the series $\sum\limits_{n=1}^{\infty} \left(\dfrac{2}{3}\right)^n$ for convergence

Answer

$\sum\limits_{n=1}^{\infty} \left(\dfrac{2}{3}\right)^n$ is geometric series with ratio $\dfrac{2}{3} < 1$ which show that is convergent series

$$s_n = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = \frac{a}{1-r} = \frac{(2/3)}{1-(2/3)} = \frac{(2/3)}{1/3} = 2$$